

# GENERALIZED POLYA-SZEGÖ INEQUALITY

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ABSTRACT. We generalize Polya-Szegö inequality to integrands depending on  $u$  and its gradient. Under minimal additional assumptions, we establish equality cases in this generalized inequality.

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## 1. INTRODUCTION

The Polya-Szegö inequality asserts that the  $L^2$  norm of the gradient of a positive function  $u$  in  $W^{1,p}(\mathbb{R}^N)$  cannot increase under Schwarz symmetrization,

$$(1.1) \quad \int_{\mathbb{R}^N} |\nabla u^*|^2 dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

The Schwarz rearrangement of  $u$  is denoted here by  $u^*$ . Inequality (1.1) has numerous applications in physics. It was first used in 1945 by G. Polya and G. Szegö to prove that the capacity of a condenser diminishes or remains unchanged by applying the process of Schwarz symmetrization (see [30]). Inequality (1.1) was also the key ingredients to show that, among all bounded bodies with fixed measure, balls have the minimal capacity (see [26, Theorem 11.17]). Finally (1.1) has also played a crucial role in the solution of the famous Choquard's conjecture (see [25]). It is heavily connected to the isoperimetric inequality and to Riesz-type rearrangement inequalities. Moreover, it turned out that (1.1) is extremely helpful in establishing the existence of ground states solutions of the nonlinear Schrödinger equation

$$(1.2) \quad \begin{cases} i\partial_t \Phi + \Delta \Phi + f(|x|, \Phi) = 0 & \text{in } \mathbb{R}^N \times (0, \infty), \\ \Phi(x, 0) = \Phi_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

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A ground state solution of equation (1.2) is a positive solution to the following associated variational problem

$$(1.3) \quad \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} F(|x|, u) dx : u \in H^1(\mathbb{R}^N), \|u\|_{L^2} = 1 \right\},$$

where  $F(|x|, s)$  is the primitive of  $f(|x|, \cdot)$  with  $F(|x|, 0) = 0$ . Inequality (1.1) together with the generalized Hardy-Littlewood inequality were crucial to prove that (1.3) admits a radial and radially decreasing solution. Furthermore, under appropriate regularity assumptions on the nonlinearity  $F$ , there exists a Lagrange multiplier  $\lambda$  such that any minimizer of (1.3) is a solution of the following semi-linear elliptic PDE

$$-\Delta u + f(|x|, u) + \lambda u = 0, \quad \text{in } \mathbb{R}^N.$$

We refer the reader to [20] for a detailed analysis. The same approach applies to the more general quasi-linear PDE

$$-\Delta_p u + f(|x|, u) + \lambda u = 0, \quad \text{in } \mathbb{R}^N.$$

where  $\Delta_p u$  means  $\operatorname{div}(|\nabla u|^{p-2} \nabla u)$ , and we can derive similar properties of ground state solutions since (1.1) extends to gradients that are in  $L^p(\mathbb{R}^N)$  in place of  $L^2(\mathbb{R}^N)$ , namely

$$(1.4) \quad \int_{\mathbb{R}^N} |\nabla u^*|^p dx \leq \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

Due to the multitude of applications in physics, rearrangement inequalities like (1.1) and (1.4) have attracted a huge number of mathematicians from the middle of the last century. Different approaches were built up to establish these inequalities such as heat-kernel methods, slicing and cut-off techniques and two-point rearrangement.

A generalization of inequality (1.4) to suitable convex integrands  $A : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,

$$(1.5) \quad \int_{\mathbb{R}^N} A(|\nabla u^*|) dx \leq \int_{\mathbb{R}^N} A(|\nabla u|) dx,$$

was first established by Almgren and Lieb (see [1]). Inequality (1.5) is important in studying the continuity and discontinuity of Schwarz symmetrization in Sobolev spaces (see e.g. [1, 11]). It also permits us to study symmetry properties of variational problems involving integrals of type  $\int_{\mathbb{R}^N} A(|\nabla u|) dx$ . Extensions of Polya-Szegö inequality to more general operators of the form

$$j(s, \xi) = b(s)A(|\xi|), \quad s \in \mathbb{R}, \xi \in \mathbb{R}^N,$$

on bounded domains have been investigated by Kawohl, Mossino and Bandle. More precisely, they proved that

$$(1.6) \quad \int_{\Omega^*} b(u^*)A(|\nabla u^*|) dx \leq \int_{\Omega} b(u)A(|\nabla u|) dx,$$

where  $\Omega^*$  denotes the ball in  $\mathbb{R}^N$  centered at the origin having the Lebesgue measure of  $\Omega$ , under suitably convexity, monotonicity and growth assumptions (see e.g. [3, 24, 29]). Numerous applications of (1.6) have been discussed in the above references. In [35],

Tahraoui claimed that a general integrand  $j(s, \xi)$  with appropriate properties can be written in the form

$$\sum_{i=1}^{\infty} b_i(s) A_i(|\xi|) + R_1(s) + R_2(\xi), \quad s \in \mathbb{R}, \xi \in \mathbb{R}^N,$$

where  $b_i$  and  $A_i$  are such that inequality (1.6) holds. However, there are some mistakes in [35] and we do not believe that this density type result holds true. Until quite recently there were no results dealing with the generalized Polya-Szegö inequality, namely

$$(1.7) \quad \int_{\Omega^*} j(u^*, |\nabla u^*|) dx \leq \int_{\Omega} j(u, |\nabla u|) dx.$$

While writing down this paper we have learned about a very recent survey by F. Brock [6] who was able to prove (1.7) under continuity, monotonicity, convexity and growth conditions.

Following a completely different approach, we prove (1.7) without requiring any growth conditions on  $j$ . As it can be easily seen it is important to drop these conditions to be able to cover some relevant applications. Our approach is based upon a suitable approximation of the Schwarz symmetrized  $u^*$  of a function  $u$ . More precisely, if  $(H_n)_{n \geq 1}$  is a dense sequence in the set of closed half spaces  $H$  containing 0 and  $u \in L^p_+(\mathbb{R}^N)$ , there exists a sequence  $(u_n)$  consisting of iterated polarizations of the  $H_n$ s which converges to  $u^*$  in  $L^p(\mathbb{R}^N)$  (see [17, 38]). On the other hand, a straightforward computation shows that

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} = \|\nabla u_0\|_{L^p(\mathbb{R}^N)} = \cdots = \|\nabla u_n\|_{L^p(\mathbb{R}^N)}, \quad \text{for all } n \in \mathbb{N}.$$

By combining these properties with the weak lower semicontinuity of the functional  $J(u) = \int j(u, |\nabla u|) dx$  enable us to conclude (see Theorem 3.1). Note that (1.5) was proved using coarea formula; however this approach does not apply to integrands depending both on  $u$  and its gradient since one has to apply simultaneously the coarea formula to  $|\nabla u|$  and to decompose  $u$  with the Layer-Cake principle.

Notice that Brock's method is based on an intermediate maximization problem and cannot yield to the establishment of equality cases. Our approximation approach was also fruitful in determining the relationship between  $u$  and  $u^*$  such that

$$(1.8) \quad \int_{\mathbb{R}^N} j(u^*, |\nabla u^*|) dx = \int_{\mathbb{R}^N} j(u, |\nabla u|) dx.$$

Indeed, under very general conditions on  $j$ , we prove that (1.8) is equivalent to

$$\int_{\mathbb{R}^N} |\nabla u^*|^p dx = \int_{\mathbb{R}^N} |\nabla u|^p dx.$$

For  $j(\xi) = |\xi|^p$ , identity cases were completely studied in the breakthrough paper of Brothers and Ziemer [10].

The paper is organized as follows.

Section 2 is dedicated to some preliminary stuff, especially the ones concerning the invariance of a class of functionals under polarization. These observations are crucial, in Section 3, to establish in a simple way the generalized Polya-Szegö inequality.

### Notations.

- (1) For  $N \in \mathbb{N}$ ,  $N \geq 1$ , we denote by  $|\cdot|$  the euclidean norm in  $\mathbb{R}^N$ .
- (2)  $\mathbb{R}_+$  (resp.  $\mathbb{R}_-$ ) is the set of positive (resp. negative) real values.
- (3)  $\mu$  denotes the Lebesgue measure in  $\mathbb{R}^N$ .
- (4)  $M(\mathbb{R}^N)$  is the set of measurable functions in  $\mathbb{R}^N$ .
- (5) For  $p > 1$  we denote by  $L^p(\mathbb{R}^N)$  the space of  $f$  in  $M(\mathbb{R}^N)$  with  $\int_{\mathbb{R}^N} |f|^p dx < \infty$ .
- (6) The norm  $(\int_{\mathbb{R}^N} |f|^p dx)^{1/p}$  in  $L^p(\mathbb{R}^N)$  is denoted by  $\|\cdot\|_p$ .
- (7) For  $p > 1$  we denote by  $W^{1,p}(\mathbb{R}^N)$  the Sobolev space of functions  $f$  in  $L^p(\mathbb{R}^N)$  having generalized partial derivatives  $D_i f$  in  $L^p(\mathbb{R}^N)$ , for  $i = 1, \dots, N$ .
- (8)  $D^{1,p}(\mathbb{R}^N)$  is the space of measurable functions whose gradient is in  $L^p(\mathbb{R}^N)$ .
- (9)  $L_+^p(\mathbb{R}^N)$  is the cone of positive functions of  $L^p(\mathbb{R}^N)$ .
- (10)  $W_+^{1,p}(\mathbb{R}^N)$  is the cone of positive functions of  $W^{1,p}(\mathbb{R}^N)$ .
- (11) For  $R > 0$ ,  $B(0, R)$  is the ball in  $\mathbb{R}^N$  centered at zero with radius  $R$ .

## 2. PRELIMINARY STUFF

In the following  $H$  will design a closed half-space of  $\mathbb{R}^N$  containing the origin,  $0_{\mathbb{R}^N} \in H$ . We denote by  $\mathcal{H}$  the set of closed half-spaces of  $\mathbb{R}^N$  containing the origin. We shall equip  $\mathcal{H}$  with a topology ensuring that  $H_n \rightarrow H$  as  $n \rightarrow \infty$  if there is a sequence of isometries  $i_n : \mathbb{R}^N \rightarrow \mathbb{R}^N$  such that  $H_n = i_n(H)$  and  $i_n$  converges to the identity as  $n \rightarrow \infty$ .

We first recall some basic notions. For more details, we refer the reader to [12].

**Definition 2.1.** A reflection  $\sigma : \mathbb{R}^N \rightarrow \mathbb{R}^N$  with respect to  $H$  is an isometry such that the following properties hold

- (1)  $\sigma \circ \sigma(x) = x$ , for all  $x \in \mathbb{R}^N$ ;
- (2) the fixed point set of  $\sigma$  separates  $\mathbb{R}^N$  in  $H$  and  $\mathbb{R}^N \setminus H$  (interchanged by  $\sigma$ );
- (3)  $|x - y| < |x - \sigma(y)|$ , for all  $x, y \in H$ .

Given  $x \in \mathbb{R}^N$ , the reflected point  $\sigma_H(x)$  will also be denoted by  $x^H$ .

**Definition 2.2.** Let  $H$  be a given half-space in  $\mathbb{R}^N$ . The two-point rearrangement (or polarization) of a nonnegative real valued function  $u : \mathbb{R}^N \rightarrow \mathbb{R}_+$  with respect to a given reflection  $\sigma_H$  (with respect to  $H$ ) is defined as

$$u^H(x) := \begin{cases} \max\{u(x), u(\sigma_H(x))\}, & \text{for } x \in H, \\ \min\{u(x), u(\sigma_H(x))\}, & \text{for } x \in \mathbb{R}^N \setminus H. \end{cases}$$

**Definition 2.3.** We say that a nonnegative measurable function  $u$  is symmetrizable if  $\mu(\{x \in \mathbb{R}^N : u(x) > t\}) < \infty$  for all  $t > 0$ . The space of symmetrizable functions is

denoted by  $F_N$  and, of course,  $L_+^p(\mathbb{R}^N) \subset F_N$ . Also, two functions  $u, v$  are said to be equimeasurable (and we shall write  $u \sim v$ ) when

$$\mu(\{x \in \mathbb{R}^N : u(x) > t\}) = \mu(\{x \in \mathbb{R}^N : v(x) > t\}),$$

for all  $t > 0$ .

**Definition 2.4.** For a given  $u$  in  $F_N$ , the Schwarz symmetrization  $u^*$  of  $u$  is the unique function with the following properties (see e.g. [19])

- (1)  $u$  and  $u^*$  are equimeasurable;
- (2)  $u^*(x) = h(|x|)$ , where  $h : (0, \infty) \rightarrow \mathbb{R}_+$  is a continuous and decreasing function.

In particular,  $u$ ,  $u^H$  and  $u^*$  are all equimeasurable functions (see e.g. [2]).

**Lemma 2.5.** Let  $u \in W_+^{1,p}(\mathbb{R}^N)$  and let  $H$  be a given half-space. Then  $u^H \in W_+^{1,p}(\mathbb{R}^N)$  and, setting

$$v(x) := u(x^H), \quad w(x) := u^H(x^H), \quad x \in \mathbb{R}^N,$$

the following facts hold:

- (1) We have

$$\begin{aligned} \nabla u^H(x) &= \begin{cases} \nabla u(x) & \text{for } x \in \{u > v\} \cap H, \\ \nabla v(x) & \text{for } x \in \{u \leq v\} \cap H, \end{cases} \\ \nabla w(x) &= \begin{cases} \nabla v(x) & \text{for } x \in \{u > v\} \cap H, \\ \nabla u(x) & \text{for } x \in \{u \leq v\} \cap H. \end{cases} \end{aligned}$$

- (2) For all  $i = 1, \dots, N$  and  $p \in (1, \infty)$ , we have

$$(2.1) \quad \|D_i u^H\|_{L^p(\mathbb{R}^N)} = \|D_i u\|_{L^p(\mathbb{R}^N)}.$$

- (3) Let  $j : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  be a Borel measurable function. Then

$$(2.2) \quad \int_{\mathbb{R}^N} j(u, |\nabla u|) dx = \int_{\mathbb{R}^N} j(u^H, |\nabla u^H|) dx,$$

provided that  $0 \in H$  and that both integrals are finite.

*Proof.* Observing that, for all  $x \in H$ , we have

$$u^H(x) = v(x) + (u(x) - v(x))^+, \quad w(x) = u(x) - (u(x) - v(x))^+,$$

in light of [26, Corollary 6.18] it follows that  $v, w$  belong to  $W_+^{1,p}(\mathbb{R}^N)$ . Assertion (1) follows by a simple direct computation. Assertion (2) follows as a consequence of assertion (1). Concerning (3), writing  $\sigma_H$  as  $\sigma_H(x) = x_0 + Rx$ , where  $R$  is an orthogonal linear transformation, taking into account that  $|\det R| = 1$  and

$$|\nabla v(x)| = |\nabla(u(\sigma_H(x)))| = |R(\nabla u(\sigma_H(x)))| = |(\nabla u)(\sigma_H(x))|,$$

we have

$$\begin{aligned}
\int_{\mathbb{R}^N} j(u, |\nabla u|) dx &= \int_H j(u, |\nabla u|) dx + \int_{\mathbb{R}^N \setminus H} j(u, |\nabla u|) dx \\
&= \int_H j(u, |\nabla u|) dx + \int_H j(u(\sigma_H(x)), |(\nabla u)(\sigma_H(x))|) dx \\
&= \int_H j(u, |\nabla u|) dx + \int_H j(v, |\nabla v|) dx.
\end{aligned}$$

In a similar fashion, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} j(u^H, |\nabla u^H|) dx &= \int_H j(u^H, |\nabla u^H|) dx + \int_H j(u^H(\sigma_H(x)), |(\nabla u^H)(\sigma_H(x))|) dx \\
&= \int_H j(u^H, |\nabla u^H|) dx + \int_H j(w, |\nabla w|) dx \\
&= \int_{\{u > v\} \cap H} j(u, |\nabla u|) dx + \int_{\{u > v\} \cap H} j(v, |\nabla v|) dx \\
&\quad + \int_{\{u \leq v\} \cap H} j(v, |\nabla v|) dx + \int_{\{u \leq v\} \cap H} j(u, |\nabla u|) dx \\
&= \int_H j(u, |\nabla u|) dx + \int_H j(v, |\nabla v|) dx,
\end{aligned}$$

which concludes the proof  $\square$

### 3. GENERALIZED POLYA-SZEGÖ INEQUALITY

The first main result of the paper is the following

**Theorem 3.1.** *Let  $\varrho : [0, \infty) \times \mathbb{R}^N \rightarrow \mathbb{R}$  be a Borel measurable function. For any function  $u \in W_+^{1,p}(\mathbb{R}^N)$ , let us set*

$$J(u) = \int_{\mathbb{R}^N} \varrho(u, \nabla u) dx.$$

*Moreover, let  $(H_n)_{n \geq 1}$  be a dense sequence in the set of closed half spaces containing  $0_{\mathbb{R}^N}$ . For  $u \in W_+^{1,p}(\mathbb{R}^N)$ , define a sequence  $(u_n)$  by setting*

$$\begin{cases} u_0 = u \\ u_{n+1} = u_n^{H_1 \dots H_{n+1}}. \end{cases}$$

*Assume that the following conditions hold:*

(1)

$$-\infty < J(u) < +\infty;$$

(2)

$$(3.1) \quad \liminf_n J(u_n) \leq J(u);$$

(3) *if  $(u_n)$  converges weakly to some  $v$  in  $W_+^{1,p}(\mathbb{R}^N)$ , then*

$$J(v) \leq \liminf_n J(u_n).$$

Then

$$J(u^*) \leq J(u).$$

*Proof.* By the (explicit) approximation results contained in [17, 38], we know that  $u_n \rightarrow u^*$  in  $L^p(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Moreover, by Lemma 2.5 applied with  $j(s, |\xi|) = |\xi|^p$ , we have

$$(3.2) \quad \|\nabla u\|_{L^p(\mathbb{R}^N)} = \|\nabla u_0\|_{L^p(\mathbb{R}^N)} = \cdots = \|\nabla u_n\|_{L^p(\mathbb{R}^N)}, \quad \text{for all } n \in \mathbb{N}.$$

In particular, up to a subsequence,  $(u_n)$  is weakly convergent to some function  $v$  in  $W^{1,p}(\mathbb{R}^N)$ . By uniqueness of the weak limit in  $L^p(\mathbb{R}^N)$  one can easily check that  $v = u^*$ , namely  $u_n \rightharpoonup u^*$  in  $W^{1,p}(\mathbb{R}^N)$ . Hence, using assumption (3) and (3.1), we have

$$(3.3) \quad J(u^*) \leq \liminf_n J(u_n) \leq J(u),$$

concluding the proof.  $\square$

**Remark 3.2.** A quite large class of functionals  $J$  which satisfy assumption (3.1) of the previous Theorem is provided by Lemma 2.5.

**Corollary 3.3.** *Let  $j : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$  be a function satisfying the following assumptions:*

- (1)  $j(\cdot, t)$  is continuous for all  $t \in [0, \infty)$ ;
- (2)  $j(s, \cdot)$  is convex for all  $s \in [0, \infty)$  and continuous at zero;
- (3)  $j(s, \cdot)$  is nondecreasing for all  $s \in [0, \infty)$ .

Then, for all function  $u \in W_+^{1,p}(\mathbb{R}^N)$  such that

$$\int_{\mathbb{R}^N} j(u, |\nabla u|) dx < \infty,$$

we have

$$\int_{\mathbb{R}^N} j(u^*, |\nabla u^*|) dx \leq \int_{\mathbb{R}^N} j(u, |\nabla u|) dx.$$

*Proof.* The assumptions on  $j$  imply that  $\{\xi \mapsto j(s, |\xi|)\}$  is convex so that the weak lower semicontinuity assumption of Theorem 3.1 holds (we refer the reader e.g. to the papers [21, 22] by A. Ioffe). Also, assumption (3.1) of Theorem 3.1 is provided by means of Lemma 2.5.  $\square$

**Remark 3.4.** In [6, Theorem 4.3], F. Brock proved Corollary 3.3 for Lipschitz functions having compact support. In order to prove the most interesting cases in the applications, the inequality has to hold for functions  $u$  in  $W_+^{1,p}(\mathbb{R}^N)$ . This forces him to assume some growth conditions of the Lagrangian  $j$ , for instance to assume that there exists a positive constant  $K$  and  $q \in [p, p^*]$  such that

$$|j(s, |\xi|)| \leq K(s^q + |\xi|^p), \quad \text{for all } s \in \mathbb{R}_+ \text{ and } \xi \in \mathbb{R}^N.$$

By our approach, instead, can include integrands such as

$$j(s, |\xi|) = \frac{1}{2}(1 + s^{2\alpha})|\xi|^p, \quad \text{for all } s \in \mathbb{R}_+ \text{ and } \xi \in \mathbb{R}^N,$$

for some  $\alpha > 0$ , which have meaningful physical applications (for instance quasi-linear Schrödinger equations, see [27] and references therein). We also stress that the approach of [6] cannot yield the establishment of equality cases (see Theorem 3.6).

**Corollary 3.5.** *Let  $m \geq 1$  and  $p_1, \dots, p_m \in (1, \infty)$ . Then*

$$\sum_{i=1}^m \int_{\mathbb{R}^N} |D_i u^*|^{p_i} dx \leq \sum_{i=1}^m \int_{\mathbb{R}^N} |D_i u|^{p_i} dx,$$

for all  $u \in \bigcap_{i=1}^m W_+^{1,p_i}(\mathbb{R}^N)$ .

*Proof.* The assertion follows by a simple combination of Theorem 3.1 with inequality (2.1) of Lemma 2.5.  $\square$

**Theorem 3.6.** *In addition to the assumptions of Theorem 3.1, assume that*

$$(3.4) \quad J(u_n) \rightarrow J(u^*) \text{ as } n \rightarrow \infty \text{ implies that } u_n \rightarrow u^* \text{ in } D^{1,p}(\mathbb{R}^N) \text{ as } n \rightarrow \infty.$$

Then

$$J(u) = J(u^*) \implies \|\nabla u\|_{L^p(\mathbb{R}^N)} = \|\nabla u^*\|_{L^p(\mathbb{R}^N)}.$$

*Proof.* Assume that  $J(u) = J(u^*)$ . Then, by assumption (3.1), we obtain

$$J(u^*) = \lim_n J(u_n) = J(u).$$

In turn, by assumption,  $u_n \rightarrow u^*$  in  $D^{1,p}(\mathbb{R}^N)$  as  $n \rightarrow \infty$ . Then, taking the limit inside equalities (3.2), we conclude the assertion.  $\square$

**Remark 3.7.** Assume that  $\{\xi \mapsto j(s, |\xi|)\}$  is strictly convex for any  $s \in \mathbb{R}_+$  and there exists  $\nu' > 0$  such that  $j(s, |\xi|) \geq \nu' |\xi|^p$  for all  $s \in \mathbb{R}_+$  and  $\xi \in \mathbb{R}^N$ . Then assumption (3.4) is fulfilled for  $J(u) = \int_{\mathbb{R}^N} j(u, |\nabla u|) dx$ . We refer to [39, Section 3].

**Remark 3.8.** Equality cases of the type  $\|\nabla u\|_{L^p(\mathbb{R}^N)} = \|\nabla u^*\|_{L^p(\mathbb{R}^N)}$  have been completely characterized in the breakthrough paper by Brothers and Ziemer [10].

Let us now set

$$M = \text{esssup}_{\mathbb{R}^N} u = \text{esssup}_{\mathbb{R}^N} u^*, \quad C^* = \{x \in \mathbb{R}^N : \nabla u^*(x) = 0\}.$$

**Corollary 3.9.** *Assume that  $\{\xi \mapsto j(s, |\xi|)\}$  is strictly convex and there exists a positive constant  $\nu'$  such that*

$$j(s, |\xi|) \geq \nu' |\xi|^p, \quad \text{for all } s \in \mathbb{R} \text{ and } \xi \in \mathbb{R}^N.$$

Moreover, assume that

$$\int_{\mathbb{R}^N} j(u, |\nabla u|) dx = \int_{\mathbb{R}^N} j(u^*, |\nabla u^*|) dx, \quad \mu(C^* \cap (u^*)^{-1}(0, M)) = 0.$$

Then there exists  $x_0 \in \mathbb{R}^N$  such that

$$u(x) = u^*(x - x_0), \quad \text{for all } x \in \mathbb{R}^N,$$

namely  $u$  is radially symmetric after a translation in  $\mathbb{R}^N$ .

*Proof.* It is sufficient to combine Theorem 3.6 with [10, Theorem 1.1].  $\square$



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